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On the stability of motion of a radiating electron

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Abstract. An integro-differential equation of motion for the radiating electron is proposed. In some special cases the stability of the solutions is investigated.

1. The equation of motion

In the classical theory of radiation reaction, charged particles are usually assumed to be point-like. This assumption leads to the deduction of the well known Lorentz–Dirac equation of motion:

$$m\dot{u}^{\nu}(\tau)-\frac{2}{3}\frac{e^2}{c^2}\left(\ddot{u}^{\nu}(\tau)+\frac{1}{c^2}\dot{u}^{\mu}(\tau)\dot{u}_{\mu}(\tau)u^{\nu}(\tau)\right)=K^{\nu}. \tag{1.1}$$

Here m means the (total) mass of the particle, $\{u^{\nu}\}\doteq\{u^0, u^n\}$ its four-velocity, τ the proper time and K^{ν} the external forces.

This equation contains run-away solutions and other unphysical properties, often discussed in the literature (cf the review article by Erber 1961). The modifications proposed hitherto are not satisfying. One of the reasons may be as follows.

In the theory of special relativity there exists no rigid body. An extended particle possesses an infinite number of internal degrees of freedom (von Laue 1911), whereas a point particle has none. So in a strictly mathematical sense equation (1.1) is not derivable from an equation of motion for an extended particle by going to the limit of vanishing extension. The limit is not well defined but shows singularities.

In order to get an improved equation of motion for the centre of mass the properties of an elastic body have to be simulated. Let us argue heuristically. Consider a particle with a radially symmetrical charge density distribution. During the acceleration process it may radiate an electromagnetic wave in a distance $c\sigma$ from the centre. Travelling only with velocity less than the speed of light the reaction cannot reach the centre before a time σ has elapsed. The magnitude of the reaction depends on the charge density of the emitting locus. Therefore the total radiation reaction on the motion of the centre should be described by an expression of the form

$$Q^{\nu}(\tau)=\frac{1}{\rho}\int_0^{\rho} f(\sigma)u^{\nu}(\tau-\sigma)\,\mathrm{d}\sigma \tag{1.2}$$

where $c\rho$ characterises the extension of the particle (diameter), and the structure function (form factor) $f(\sigma)$ takes into account the charge distribution (cf footnote to equation (1.3)).

Denoting the electromagnetic inertia by m_e and if present the mechanical inertia by m_m the following equation of motion is suggested:

$$m_m\dot{u}^\nu(\tau)+m_e\Big(\dot{Q}^\nu(\tau)-\frac{1}{c^2}\dot{Q}^\mu(\tau)u_\mu(\tau)u^\nu(\tau)\Big)=K^\nu. \tag{1.3}$$

The third left-hand term is added in order to make the system of differential equations consistent. The four-force and the four-velocity are mutually orthogonal in the Minkowski space. (The foundation of equation (1.3) on the Maxwell–Lorentz theory is far beyond the scope of this paper. It turns out to be rather lengthy and sophisticated and will be published elsewhere. Nevertheless it is easy to see that (1.3) has something to do with radiation reaction: in the limit $\rho\to 0$ (cf (1.5)) one arrives at the Lorentz–Dirac equation (1.1).

The structure of (1.3) can be made plausible by sketching some of the first steps of a derivation from the basic equations of electrodynamics. Starting from the equation of motion $m_m\dot{u}^\nu(\tau)=K^\nu+K_s^\nu$ the self-force on the particle is given by the Lorentz force density

$$\begin{aligned}K_s^\mu &=\frac{u^0}{c}\int s_\nu F_s^{\nu\mu}\,d^3x\\ &=\frac{u^0}{c}\int s_\nu\frac{\partial}{\partial x_\mu}A_s^\nu\,d^3x-\frac{u^0}{c}\int s_\nu\frac{\partial}{\partial x_\nu}A^\mu\,d^3x=\mathcal{K}_1+\mathcal{K}_2.\end{aligned}$$

Substitute the self-field $F_s^{\nu\mu}$ by the retarded potentials, e.g. in $\mathcal{K}_2$:

$$\begin{aligned}\mathcal{K}_2^\mu &=-\frac{u^0}{c}\int\frac{\partial}{\partial x^\nu}(s^\nu A_s^\mu)\,d^3x\\ &=-\frac{u^0}{c}\frac{\partial}{\partial x^0}\int s^0(x^0,x^n)D_{\rm ret}(x^0-\hat{x}^0,x^n-\hat{x}^n)s^\mu(\hat{x}^0,\hat{x}^n)\,d^3x\,d^4\hat{x}.\end{aligned}$$

The non-relativistic limit of the space-like components is

$$\begin{aligned}\mathcal{K}_2^n &\approx-\frac{\partial}{\partial t}\frac{1}{c}\int q(x^r)D_{\rm ret}(c(t-\hat{t}),x^n-\hat{x}^n)q(\hat{x}^m)\,d^3x\,d^3\hat{x}\,v^n(\hat{t})\,d\hat{t}\\ &\equiv-\frac{\partial}{\partial t}\int_{t-\rho}^{t}M(t-\hat{t})v^n(\hat{t})\,d\hat{t}\end{aligned}$$

q is the charge density in the rest frame. From the properties of q and $D_{\rm ret}$ it follows $M(\sigma)=0$ if $\sigma>\rho$ with $c\rho=$ diameter of the particle. The relativistic generalisation

$$\mathcal{K}_2^\mu\approx-\frac{\partial}{\partial\tau}\int_0^\rho M(\sigma)u^\mu(\tau-\sigma)\,d\sigma$$

is part of (not identical with) the second term on the left-hand side of equation (1.3) and has the same structure with respect to (1.2).

The discussion of $\mathcal{K}_1^\mu$ is much more complicated and has to be omitted here. But it remains true, that $f(\sigma)$ is a bilinear form of the charge density of the particle.)

Only the product $m_ef(\sigma)$ enters into equation (1.3). To make m_e definite, we choose the normalisation condition

$$\frac{1}{\rho}\int_0^\rho f(\sigma)\,d\sigma=1. \tag{1.4}$$

The connection of the parameters with the experimental data is established by comparing (1.1) and (1.3) after letting $\rho \to 0$ (though this is mathematically not justified):

$$m = m_{\mathrm{m}} + m_{\mathrm{e}}$$
$$\frac{2}{3}\frac{e^2}{c^2} = m_{\mathrm{e}} \frac{1}{\rho} \int_0^{\rho} \sigma f(\sigma)\,\mathrm{d}\sigma. \tag{1.5}$$

The quantity proportional to $\dot{Q}^{\mu} u_{\mu}$ corresponds to the radiated energy. It must be negative semi-definite for all particle motions

$$\begin{aligned} 0 &\geq \dot{Q}^{\mu}(\tau) u_{\mu}(\tau) \\ &= -\frac{1}{\rho} \int_0^{\rho} f(\sigma) u_{\mu}(\tau) \frac{\mathrm{d}}{\mathrm{d}\sigma}(u^{\mu}(\tau-\sigma) - u^{\mu}(\tau))\,\mathrm{d}\sigma \\ &= -\frac{1}{\rho} f(\rho)(u_{\mu}(\tau) u^{\mu}(\tau-\rho) - c^2) + \frac{1}{\rho}\int_0^{\rho} f'(\sigma)(u_{\mu}(\tau)u^{\mu}(\tau-\sigma) - c^2)\,\mathrm{d}\sigma. \end{aligned} \tag{1.6}$$

With regard to $u_{\mu}(\tau)u^{\mu}(\tau-\sigma) \geq c^2$ this is satisfied if

$$\begin{aligned} &f(\rho) \geq 0 \\ &f'(\sigma) \leq 0 \qquad \text{if } 0 \leq \sigma \leq \rho. \end{aligned} \tag{1.7}$$

These conditions imply

$$f(\sigma) \geq 0 \qquad \text{if } 0 \leq \sigma \leq \rho \tag{1.8}$$

and with respect to (1.2) and (1.4)

$$Q^0(\tau) = \frac{1}{\rho} \int_0^{\rho} f(\sigma) u^0(\tau-\sigma)\,\mathrm{d}\sigma \geq c. \tag{1.9}$$

Q^{μ} is a time-like four-vector, because $Q^0 \geq c > 0$ is true in every Lorentz frame.

The energy proportional to $\dot{u}^0$ contained in the Schott term

$$-\frac{2}{3}\frac{e^2}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\dot{u}^0(\tau)$$

of equation (1.1) is indefinite and unbounded from below. It gives rise to the unwanted self-accelerations and run-away solutions. On the contrary, the analogue term $m_{\mathrm{e}} Q^0$ in equation (1.3) is positive definite and the particle is prevented from self-accelerations (Petzold and Sorg 1977).

The new equation (1.3) is a functional differential equation. The initial value problem is determined not only by the data of the particle at a fixed time τ_0 but by the trajectory section in a time interval $\tau_0 - \rho \leq \tau \leq \tau_0$. So the infinitely many degrees of freedom of extended bodies may be simulated in this manner.

It will be shown that equation (1.3) admits damped oscillations, especially in the neighbourhood of stable solutions. The '*zitterbewegungen*' of the centre of mass simulate the vibrations of elastic bodies. Internal energy is radiated away and the oscillations are damped. The larger m_{m} the more sluggish the system and the damping is increasing. The damping constant is not bounded in the limit $m_{\mathrm{m}} \to \infty$ (cf § 3 and Petzold and Sorg 1977).

2. Radiationless motions

It is well known that an accelerated charge generally radiates. But there is an old controversy. Does a charged particle radiate or not if it is accelerated constantly? (cf Fulton and Rohrlich 1960, and references therein). For this reason we want to answer the question: provided that the new equation of motion (1.3) is correct, are there radiationless trajectories with non-constant velocities? The vanishing of the radiation implies $\dot{Q}^\mu u_\mu = 0$. Because $u^\mu_{(1)}u_{(2)\mu} = c^2$ if and only if $u^\mu_{(1)} = u^\mu_{(2)}$, one infers from (1.6) and (1.7) that

$$\begin{aligned} u^\mu(\tau-\rho) &= u^\mu(\tau) \qquad && \text{if } f(\rho)\neq 0 \\ u^\mu(\tau-\sigma) &= u^\mu(\tau) \qquad && \text{if } f'(\sigma)\neq 0 \text{ and } \sigma\in[0,\rho]. \end{aligned} \tag{2.1}$$

In a similar way to (1.6) one obtains from (1.2)

$$\dot{Q}^\mu(\tau) = -\frac{1}{\rho} f(\rho)(u^\mu(\tau-\rho)-u^\mu(\tau)) + \frac{1}{\rho}\int_0^\rho f'(\sigma)(u^\mu(\tau-\sigma)-u^\mu(\tau))\,\mathrm{d}\sigma. \tag{2.2}$$

Now it is easy to see that (2.1) implies $\dot{Q}^\mu(\tau) = 0$. If $\Lambda_{\sigma\in[0,\rho]} f'(\sigma)\neq 0$, then radiationless trajectories exist only if $u^\mu(\tau-\sigma) = u^\mu(\tau)$, i.e. u^μ is constant or K^ν is zero according to (1.3), that means, only the force-free motion does not radiate.

Consequently charged particles radiate when they are accelerated by a constant force. They are retarded relative to a non-radiating (neutral) particle. Free falling in a constant gravitational field, they do not represent a flat inertial system in the sense of Einstein's principle of equivalence.

We see this from a simple calculation. We consider the one-dimensional motion in a constant electric field. The relevant component of the Lorentz force $eu_\mu F^{\mu\nu} = K^\nu$ is $K^1 = eu^0E$. We make the *ansatz*

$$u^0 = c\cosh\omega(\tau), \qquad u^1 = c\sinh\omega(\tau), \qquad u^2 = u^3 = 0$$

and introduce this into equation (1.3):

$$m_\mathrm{m}\dot{\omega}(\tau) + m_\mathrm{e}\frac{1}{\rho}\int_0^\rho f(\sigma)\dot{\omega}(\tau-\sigma)\cosh(\omega(\tau)-\omega(\tau-\sigma))\,\mathrm{d}\sigma = \frac{e}{c}E.$$

If the four-acceleration is constant, $\dot{\omega} = b/c$, we obtain the condition

$$eE = \left(m_\mathrm{m} + m_\mathrm{e}\frac{1}{\rho}\int_0^\rho f(\sigma)\cosh(b\sigma/c)\,\mathrm{d}\sigma\right)b.$$

Hence the effective acceleration b is less than $eE/(m_\mathrm{m}+m_\mathrm{e})$, which is the acceleration of a non-radiating particle of the same total mass.

Specifying the form factor $f(\sigma)\equiv 1 (f'(\sigma)=0)$ in (1.3) a modified Caldirola equation, i.e. a differential equation with delay, is obtained (Petzold and Sorg 1977). Then all motions of periodicity ρ created by appropriate forces are radiationless. According to equation (1.3) the inertia is given then by the mechanical mass m_m only. Those trajectories have nothing to do with stationary states in quantum mechanics. These occur in binding potentials only. On the contrary, the above mentioned trajectories arise even in constant magnetic fields in the form of circular orbits. This example was numerically computed and discussed by Heudorfer and Sorg (1977a). Heudorfer and Sorg (1977b) and Sorg (1977) also treated the one-dimensional motion in a regularised Coulomb potential. The analysis of circular orbits in a Coulomb potential (the trivial calculations are omitted here) shows that the non-radiating condition does not correspond to Bohr's quantisation rules. No lower bound exists for the energy, no accumulation of energy states at the ionisation continuum is observed. Moreover, the periodical radiationless trajectories are not stable. The condition of periodicity is satisfied only by some special initial values only. Initial values of lower energy violate this condition. The particle must radiate and its path diverges more and more from the former. In addition, an instability occurs in the structure of the particle. A slight modification like $f \to \bar{f} = (1-\epsilon\sigma)/(1-\frac{1}{2}\epsilon\rho)$ so that $\bar{f}'$ is very small but not zero, destroys the radiationless motions, because now only u^μ being a constant means no radiation as pointed out above. We think that the special structure function $f \equiv 1$ is not very interesting. (The case $m_m = 0$ had been investigated by Caldirola (1956). It admits undamped oscillations even for force-free motions.)

3. Energy conservation and Lyapunov functionals

The external forces acting on charged particles are of electromagnetic origin:

$$K^\nu = eu_\mu F^{\mu\nu} = eu_\mu (A^{\mu|\nu} - A^{\nu|\mu}). \tag{3.1}$$

For time-independent fields $A^{\mu|0} \equiv \partial A^\mu/\partial x^0 = 0$ the energy conservation has the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_m c u^0(\tau) + m_e c Q^0(\tau) + eA^0(x^n(\tau))) = \frac{1}{c} m_e \dot{Q}^\mu(\tau) u_\mu(\tau) u^0(\tau). \tag{3.2}$$

The left-hand expression describes the variation of the energy E in time, composed of kinetic and potential energy terms

$$\mathscr{E} = m_m c u^0(\tau) + m_e c Q^0(\tau) + eA^0(\tau). \tag{3.3}$$

The right-hand side of (3.2) gives the energy radiated away per unit time. This term is negative, so that the energy $\mathscr{E}$ decreases in time.

Provided that the potential is positive, then $\mathscr{E}$ is positive and yields a Lyapunov functional. Hence $\lim_{\tau\to\infty} \dot{Q}^\mu u_\mu = 0$. The particle asymptotically approaches a trajectory, which is radiationless but not necessarily stable (as pointed out in § 2 in the case of $f \equiv 1$).

If $\Lambda_{\sigma\in[0,\rho]} f'(\sigma) \neq 0$, then (cf § 2) $\dot{Q}^\mu \to 0$ and u^μ tends to a constant and consequently $K^\mu \to 0$ according to (1.3). The force equals the (negative) gradient of the potential. Therefore the particle comes to rest at an extreme point of the potential. The stationary point is stable, if it is a minimum of the potential.

Let us see for $K^\mu = 0$ how a radiationless path is approached. Because u^μ tends to a constant we may choose a Lorentz frame in which the particle asymptotically comes to rest. For the sake of simplicity put

$$f(\sigma) = 2(\rho - \sigma)/\rho$$

i.e.

$$f(\rho) = 0, \qquad f'(\sigma) = \text{constant} = -2/\rho \tag{3.4}$$

and consider the one-dimensional motion $u^1 = u$, $u^2 = u^3 = 0$. With regard to (1.6) and (2.2) the equation of motion (1.3) becomes

$$m_m\ddot{u}(\tau)+m_e\frac{1}{\rho}\int_0^\rho f'(\sigma)(u(\tau-\sigma)-u(\tau))\,d\sigma$$

$$=m_e\frac{1}{\rho}\int_0^\rho f'(\sigma)(u_\mu(\tau)u^\mu(\tau-\sigma)-c^2)\,d\sigma u(\tau). \tag{3.5}$$

In view of $u\to 0$ we may asymptotically use the linearised equation

$$\lambda\dot{u}(\tau)-\frac{1}{\rho^2}\int_0^\rho (u(\tau-\sigma)-u(\tau))\,d\sigma=0 \qquad \lambda=\frac{m_m}{2m_e}. \tag{3.6}$$

This equation can be solved by Laplace transformation. But for our purposes it is sufficient to consider damped oscillations

$$u(\tau)\propto\exp(\gamma\tau/\rho). \tag{3.7}$$

The negative real part $-\gamma_1/\rho$ of the complex frequency $\gamma/\rho=(\gamma_1+i\gamma_2)/\rho$ is the damping constant, the imaginary part γ_2/ρ is the frequency of the oscillation.

From (3.6) and (3.7) we derive the equation for γ:

$$1-\gamma-e^{-\gamma}=\lambda\gamma^2. \tag{3.8}$$

Apart from the trivial solution $\gamma=0$ a discussion of (3.8) shows the following. We have $-\gamma_1>0$ for every value of $m_m>0$. The weakest damping is characterised by $-\gamma_1>2$. The system is damped very strongly even in the case of missing mechanical inertia $m_m=0$. Adding the mechanical mass the damping increases and the stability is improved.

Putting (3.8) in the form

$$[e^{-\gamma}-(1-\gamma+\tfrac{1}{2}\gamma^2)]=-(\lambda+\tfrac{1}{2})$$

it is easily seen, that $m_m\to\infty$ implies $|\gamma|\to\infty$. Because of $d\gamma_2/d\lambda\gtrless 0$ for $\gamma_2\lesseqgtr 0$ (which is easily inferred from (3.8)) one concludes $\lim_{m_m\to\infty}(-\gamma_1)\to\infty$.

Thus the free motion is asymptotically stable, where we remark that simple stability is a consequence of (3.2) and (1.9):

$$m_m cu^0(\tau_0)+m_e cQ^0(\tau_0)$$

$$\geqslant m_m cu^0(\tau)+m_e cQ^0(\tau)\geqslant m_m cu^0(\tau)+m_e c^2 \qquad \text{for } \tau\geqslant\tau_0$$

i.e. $\Lambda_{\tau\geqslant\tau_0}-u^n(\tau)u_n(\tau)$ is bounded.

4. Stability behaviour of a special equation of motion for $f(\sigma)\equiv 1$

In spite of the purpose being to investigate the stability properties of the integro-differential equation (1.3) in a further paper, it seems to be methodically instructive to treat here in some detail the Lyapunov stability of the solutions of the equation of motion corresponding to the case $f(\sigma)\equiv 1$ (Petzold and Sorg 1977).

A comparison of the paper by Petzold and Sorg (1977) and § 3 shows that the introduction of $f(\sigma)\not\equiv 1$ causes an improvement of the local stability for the free motion.

Let us consider the following special case of the one-dimensional motion (cf Petzold and Sorg 1977) for $f(\sigma)\equiv 1$ and

$$\begin{aligned} &\left\{\frac{1}{c}u^{\nu}(\tau)\right\}=\{\cosh\omega(\tau),\sinh\omega(\tau),0,0\} \\ &\dot{\omega}(\tau)+\alpha\sinh\Delta\omega(\tau)=k(\tau) \end{aligned} \tag{4.1}$$

where

$$\alpha=m_e/\rho m_m, \qquad \rho=\text{constant}>0$$

$$\Delta\omega(\tau)\equiv\omega(\tau)-\omega(\tau-\rho).$$

Let $M\equiv(\omega_0(\tau)+\text{constant})$ be the set of solutions of (4.1) which we want to examine for stability, where $\omega_0(\tau)$ is a solution of (4.1). We reformulate the problem such that the stability of the set of solutions $\tilde{M}\equiv\text{constant}$ of the following equation of motion has to be investigated:

$$\dot{v}(\tau)=-a\sinh(\Delta v(\tau)+\Delta\omega_0(\tau))+l(\tau) \tag{4.2}$$

where

$$v(\tau)\equiv\omega(\tau)-\omega_0(\tau)$$

$$l(\tau)\equiv k(\tau)-\dot{\omega}_0(\tau)=\alpha\sinh\Delta\omega_0(\tau).$$

Let us abbreviate the right-hand side of (4.2):

$$g[v(\tau),v(\tau-\rho);\tau]\equiv-\alpha\sinh(\Delta v(\tau)+\Delta\omega_0(\tau))+l(\tau).$$

We assume existence (boundedness and continuity) and uniqueness of the solution $v(\tau)$ of (4.2) for given initial value $\lim_{\tau\to\tau_0+0}v(\tau)=v^{(0)}$ and given forerunning function $v(\tau)=\phi(\tau)$ for $\tau_0-\rho\leqslant\tau\leqslant\tau_0$.

First, we use the following special case of a theorem of Razumikhin (1956) (cf also Hahn 1959).

Theorem. Let the equation

$$\dot{x}(\tau)=f(x(\tau),x(\tau-\rho);\tau)$$

be given with

$$f(0,0;\tau)\equiv 0 \qquad \text{and} \qquad \rho=\text{constant}>0,$$

satisfying the above mentioned existence and uniqueness properties. The null solution of this equation is stable if there exists a positive-definite function $L(y;\tau)$ with the following property.

In the case of $L[x(s);s]\leqslant L[x(\tau);\tau]$ for all $s\in[\tau-\rho,\tau]$ along a solution the functional

$$\frac{\mathrm{d}L[x(\tau);\tau]}{\mathrm{d}\tau}\equiv U[x(\tau),x(\tau-\rho);\tau]$$

is not positive.

If furthermore L is decrescent† and U is negative-definite with respect to the above conditions, then the null solution is asymptotically stable.

† We refer to the usual definition that a scalar function $W(x;\tau)$, $x\in R$, is decrescent in R if a positive-definite function $V(x)$ exists such that, for all $x\in R$ and all τ, $|W(x;\tau)|\leqslant V(x)$.

For (4.2) we choose the positive-definite decrescent function $L(v)=v^2$ and get

$$\frac{\mathrm{d}L}{\mathrm{d}\tau}=2v(\tau)g[v(\tau), v(\tau-\rho); \tau]$$

where

$$g[v(\tau), v(\tau-\rho); \tau]$$
$$=-\alpha[\cosh \Delta\omega_0(\tau) \sinh \Delta v(\tau)+\sinh \Delta\omega_0(\tau)(\cosh \Delta v(\tau)-1)].$$

The properties of the hyperbolic functions allow the estimation

$$\frac{g(x, y; \tau)}{x}\begin{cases}\leqslant 0 & \text{for } |y|\leqslant|x| \\ =0 & \Leftrightarrow x=y\end{cases}$$

for $x\neq 0$.

In the case of $L[v(\tau-\rho)]\leqslant L[v(\tau)]$, i.e. $|v(\tau-\rho)|\leqslant|v(\tau)|$, we have therefore $\mathrm{d}L[v(\tau)]/\mathrm{d}\tau\leqslant 0$. Consequently, the stability condition of the theorem is certainly satisfied, i.e. $\tilde{M}$ is stable.

We add that $v=0$ can be mapped on any v equal to a constant by a Lorentz transformation.

Now we show that a trajectory of (4.2) which is contained in a set $M_T\equiv\{v|\Delta v(\tau)\equiv 0 \text{ for any } \tau\geqslant T\}$, $T\geqslant\tau_0$, tends to a constant.

For a trajectory $\hat{v}\in M_T$, $T\geqslant\tau_0$, of (4.2) we have $\dot{\hat{v}}\equiv 0$ for any $\tau\geqslant T$ and, successively, by differentiation of (4.2) $\hat{v}^{(n)}(\tau)\equiv 0$ for any $\tau\geqslant T+(n-1)\rho$, $n=1, 2, 3, \ldots$, such that $\lim_{\tau\to\infty}\hat{v}^{(n)}(\tau)=0$ for all $n=1, 2, 3\ldots$. This proves the proposition.

For a trajectory v of (4.2) with $v(\tau)\neq 0$ and $\Delta v(\tau)\neq 0$ at time τ the condition $L[v(s)]\leqslant L[v(\tau)]$, $\tau-\rho\leqslant s\leqslant\tau$ of the theorem yields that $\mathrm{d}L[v(\tau)]/\mathrm{d}\tau$ is negative.

From the above-proved stability and the proposition just settled about M_T we conclude that the set $\tilde{M}$ of solutions of (4.2) (and M of (4.1), respectively) is asymptotically stable generally. We remark that this result refers to cases not covered by the considerations in § 3, e.g. constant accelerations.

Appendix. Conjecture on limitations of stability

The equation of motion (1.3) admits solutions where, in spite of radiation, a charged particle is more accelerated than an uncharged one, which possesses an equal mass but does not radiate and is moving in the same external force field: $(m_\mathrm{m}+m_\mathrm{e})\dot{u}^\nu=K^\nu$.

From a mathematical point of view this is easy to see: according to (1.3) $\dot{u}^\nu(\tau)$ is determined not only by the force but the forerunning behaviour of $u^\nu(\tau-\sigma)$ for $0\leqslant\sigma\leqslant\rho$ also enters. For further discussion write (1.3) in the following form:

$$(m_\mathrm{m}+m_\mathrm{e})\dot{u}^\nu(\tau)$$
$$=K^\nu+m_\mathrm{e}\frac{1}{\rho}\int_0^\rho f(\sigma)(\dot{u}^\nu(\tau)-\dot{u}^\nu(\tau-\sigma))\,\mathrm{d}\sigma+\frac{1}{c^2}m_\mathrm{e}\dot{Q}^\mu(\tau)u_\mu(\tau)u^\nu(\tau).$$

If $\dot{u}^n(\tau-\sigma)<\dot{u}^n(\tau)$, then the second term on the right-hand side works like an additional accelerating force, meanwhile the radiation reaction force $\dot{Q}^\mu u_\mu u^n/c^2$ vanishes in the momentary rest frame of the particle. This proves the statement.

Forces increasing along the trajectory may accelerate the particle in such a manner that $\dot{u}^n(\tau-\sigma)<\dot{u}^n(\tau)$ is fulfilled even on a finite part of the path. This was observed by Sorg (1977) along a one-dimensional motion in a Coulomb potential in the case of $f\equiv 1$.

This phenomenon is intelligible by considering a model of extended charged particles. A force accelerating the front more than the back stretches the charge distribution. The Coulomb energy is lowered and the escaping energy may additionally accelerate the centre.

The more the charge density is concentrated the greater the mechanical stress has to be to maintain the equilibrium. $f'(\sigma)$ may be considered as a measure of the charge concentration and hence of the stress, too. With increasing stress the body becomes more rigid and it is more difficult to stretch it. It is supposed that in our theory the dilatation of the particle is pictured by the expression $\dot{u}^n(\tau)-\dot{u}^n(\tau-\sigma)$.

The non-existence of rigid bodies implies the boundedness of the stress tensor. Hence forces increasing too rapidly should destroy the particle. It may happen that in the presence of such external forces the existence of solutions of equation (1.3) is no longer guaranteed in general.

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